

ME 6125

Mechanics of Viscous Fluid

Lecture 1

Dr. Abdul Aziz Shuvo

Assistant Professor

Department of Mechanical Engineering, BUET

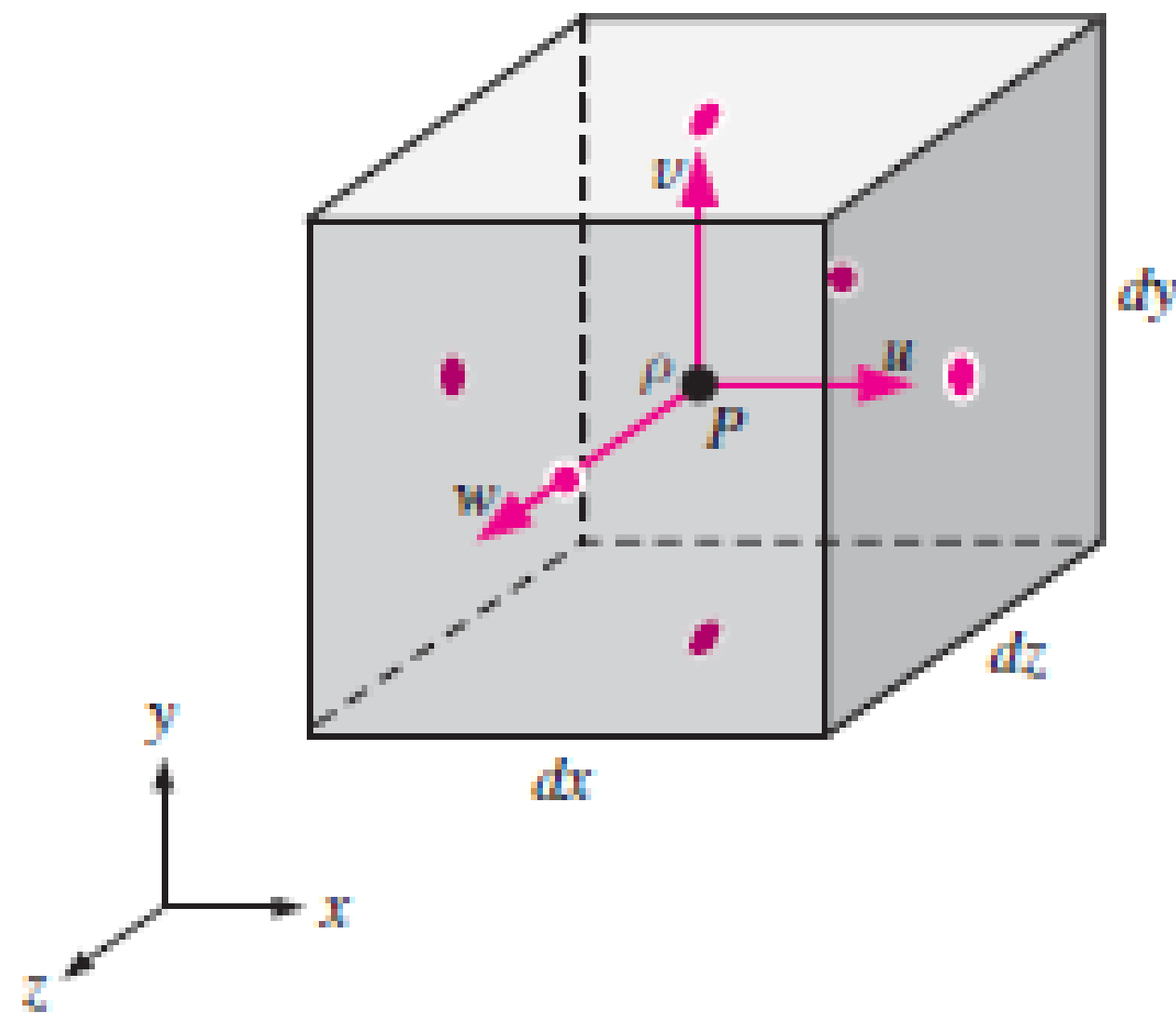


Email: shuvo@me.buet.ac.bd



IMPACT Lab
Interfacial Multiscale¹
Physics and Transport

The conservation equations – mass



Center of right face:

$$(\rho u)_{\text{center of right face}} \cong \rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2}$$

Center of left face:

$$(\rho u)_{\text{center of left face}} \cong \rho u - \frac{\partial(\rho u)}{\partial x} \frac{dx}{2}$$

Center of front face:

$$(\rho w)_{\text{center of front face}} \cong \rho w + \frac{\partial(\rho w)}{\partial z} \frac{dz}{2}$$

Center of rear face:

$$(\rho w)_{\text{center of rear face}} \cong \rho w - \frac{\partial(\rho w)}{\partial z} \frac{dz}{2}$$

Center of top face:

$$(\rho v)_{\text{center of top face}} \cong \rho v + \frac{\partial(\rho v)}{\partial y} \frac{dy}{2}$$

Center of bottom face:

$$(\rho v)_{\text{center of bottom face}} \cong \rho v - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2}$$



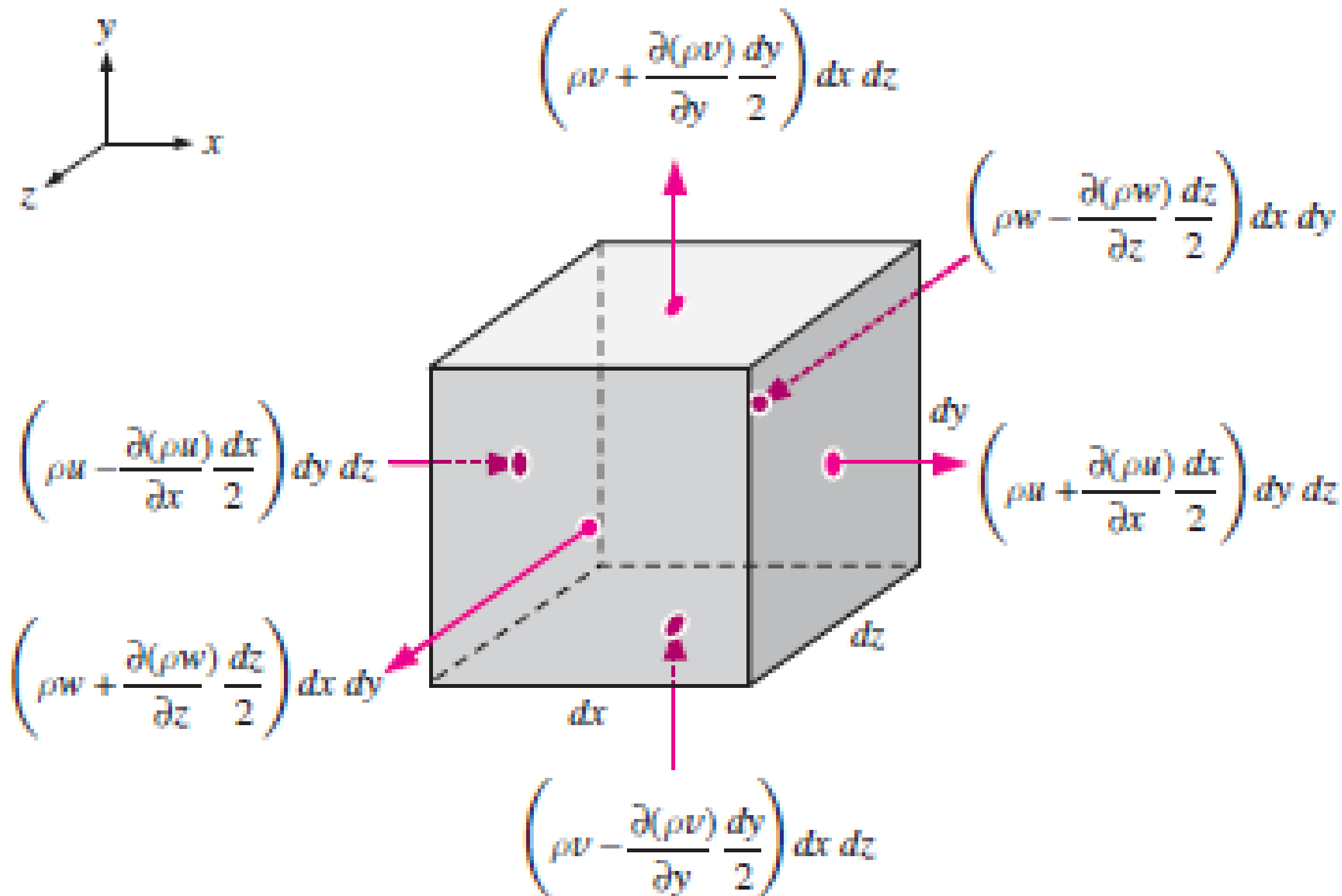
The conservation equations – mass, cont'd

Rate of change of mass within CV:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV \equiv \frac{\partial \rho}{\partial t} dx dy dz$$

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \equiv \underbrace{\left(\rho u - \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right) dy dz}_{\text{left face}} + \underbrace{\left(\rho v - \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \right) dx dz}_{\text{bottom face}} + \underbrace{\left(\rho w - \frac{\partial(\rho w)}{\partial z} \frac{dz}{2} \right) dx dy}_{\text{rear face}}$$



Net mass flow rate out of CV:

$$\sum_{out} \dot{m} \equiv \underbrace{\left(\rho u + \frac{\partial(\rho u)}{\partial x} \frac{dx}{2} \right) dy dz}_{\text{right face}} + \underbrace{\left(\rho v + \frac{\partial(\rho v)}{\partial y} \frac{dy}{2} \right) dx dz}_{\text{top face}} + \underbrace{\left(\rho w + \frac{\partial(\rho w)}{\partial z} \frac{dz}{2} \right) dx dy}_{\text{front face}}$$

$$\frac{\partial \rho}{\partial t} dx dy dz = - \frac{\partial(\rho u)}{\partial x} dx dy dz - \frac{\partial(\rho v)}{\partial y} dx dy dz - \frac{\partial(\rho w)}{\partial z} dx dy dz$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$



The conservation equations – mass, cont'd

The previous formulation can be easily extended to a 3-D system as follows:

Scalar form: $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$ where, $\vec{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

Vector form: $\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0}$ $\rightarrow$ General form of continuity

It is of interest to us to know the mass conservation equation for the following cases:

Incompressible flow: where, velocity and density gradients are such that. $\frac{D\rho}{Dt} \ll \rho \nabla \cdot \vec{V}$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

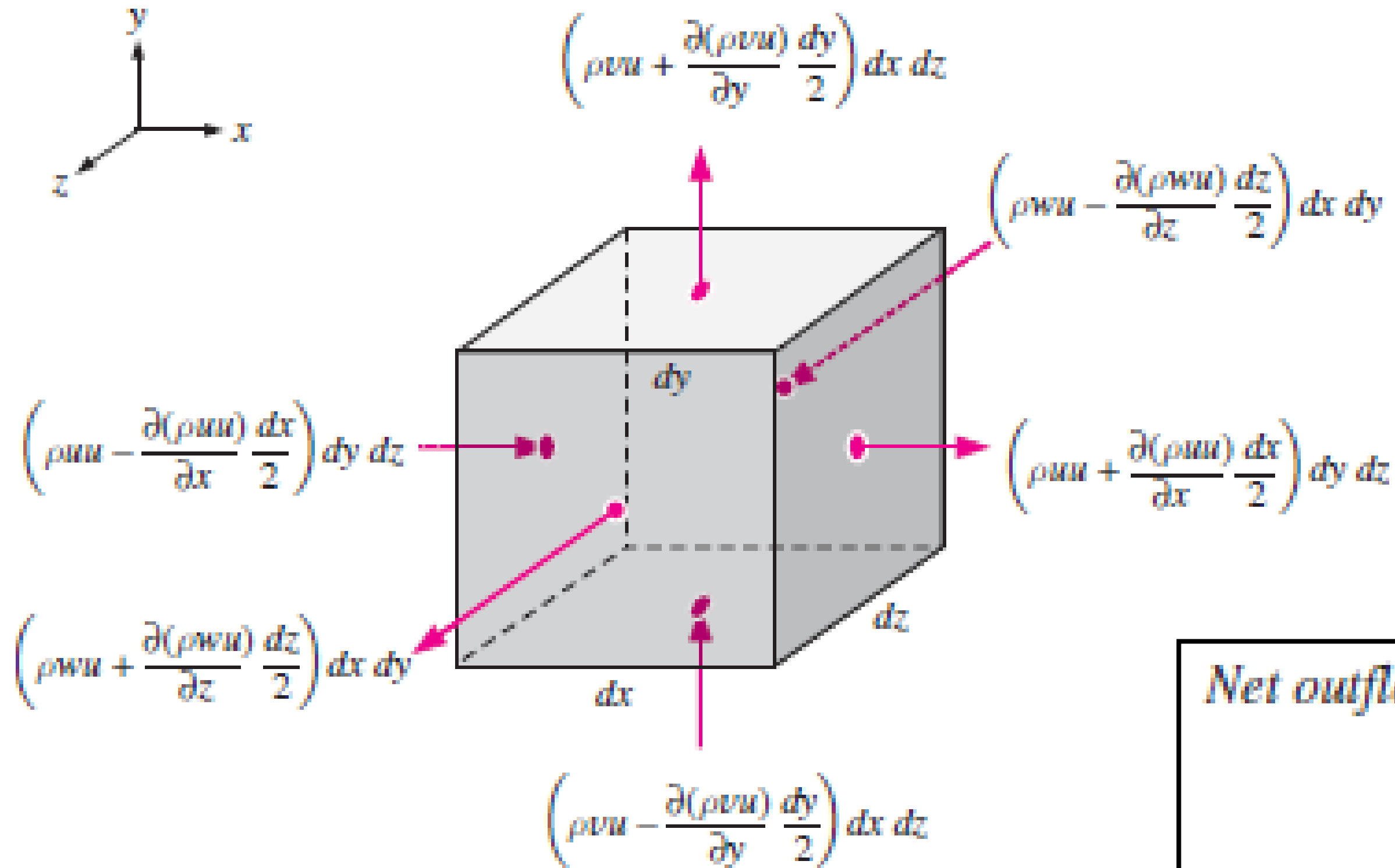
Cylindrical coordinates: $\frac{1}{r} \frac{\partial(rV_r)}{\partial r} + \frac{1}{r} \frac{\partial(V_\theta)}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$ where, $\vec{V} = V_r \mathbf{e}_r + V_\theta \mathbf{e}_\theta + V_z \mathbf{e}_z$

Spherical coordinates: $\frac{1}{r^2} \frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{\sin \phi} \frac{\partial(V_\phi \sin \phi)}{\partial \phi} + \frac{1}{\sin \theta} \frac{\partial V_\theta}{\partial \theta} = 0$
where, $\vec{V} = V_r \mathbf{e}_r + V_\phi \mathbf{e}_\phi + V_\theta \mathbf{e}_\theta$



The conservation equations – momentum

Let's consider x-component of the momentum conservation:



$$\frac{\partial (Mu)_{cv}}{\partial t} = \sum_{in} \dot{m}u - \sum_{out} \dot{m}u$$

$$\int_{cv} \frac{\partial}{\partial t} (\rho u) dV \equiv \frac{\partial}{\partial t} (\rho u) dx dy dz$$

Net outflow of x-momentum through the control surface:

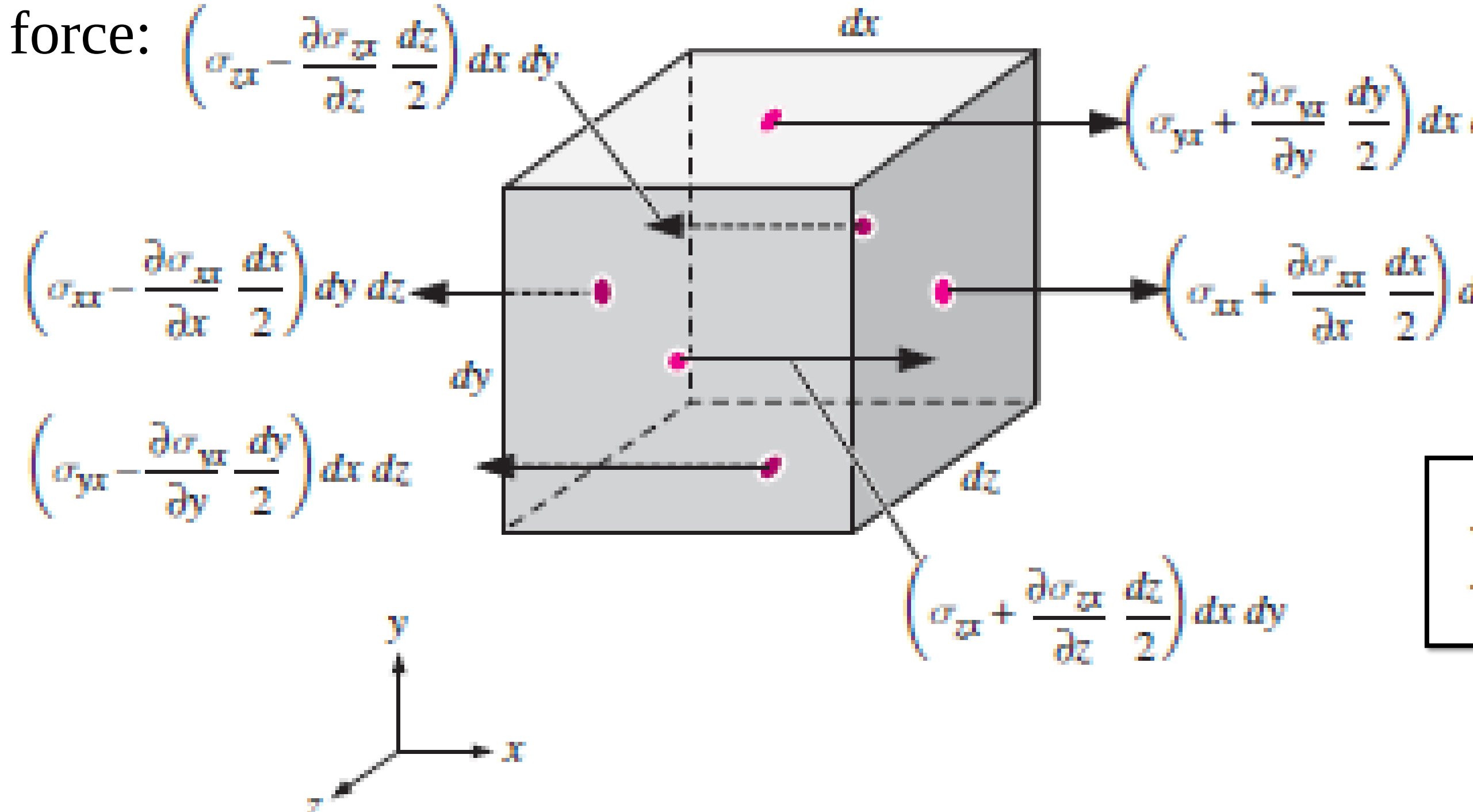
$$\sum_{out} \beta \dot{m}u - \sum_{in} \beta \dot{m}u \equiv \left(\frac{\partial}{\partial x} (\rho uu) + \frac{\partial}{\partial y} (\rho vu) + \frac{\partial}{\partial z} (\rho wu) \right) dx dy dz$$



The conservation equations – momentum, cont'd

- Net forces: (i) Body forces, (ii) Surface forces
- In x-direction, the body force (assuming only gravity) on the CV is

$$\sum F_{x, \text{body}} = \sum F_{x, \text{gravity}} \cong \rho g_x dx dy dz$$

- Surface force:
- 
- Diagram illustrating the surface forces on a control volume (CV) element in the x-direction. The CV is a rectangular prism with dimensions dx , dy , and dz . The forces are shown acting on the faces of the CV:
- Front face (yz-plane): $\left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x} \frac{dx}{2} \right) dy dz$
 - Back face (yz-plane): $\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \frac{dx}{2} \right) dy dz$
 - Right face (xz-plane): $\left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} \frac{dy}{2} \right) dx dz$
 - Left face (xz-plane): $\left(\sigma_{yx} - \frac{\partial \sigma_{yx}}{\partial y} \frac{dy}{2} \right) dx dz$
 - Top face (xy-plane): $\left(\sigma_{zx} - \frac{\partial \sigma_{zx}}{\partial z} \frac{dz}{2} \right) dx dy$
 - Bottom face (xy-plane): $\left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} \frac{dz}{2} \right) dx dy$

$$\sum F_{x, \text{surface}} \cong \left(\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{yx} + \frac{\partial}{\partial z} \sigma_{zx} \right) dx dy dz$$



The conservation equations – momentum, cont'd

- X-momentum equation:

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} = \rho g_x + \frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \sigma_{yx} + \frac{\partial}{\partial z} \sigma_{zx}$$

- Y-momentum equation:

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} = \rho g_y + \frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{zy}$$

- Z-momentum equation:

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} = \rho g_z + \frac{\partial}{\partial x} \sigma_{xz} + \frac{\partial}{\partial y} \sigma_{yz} + \frac{\partial}{\partial z} \sigma_{zz}$$



The conservation equations – momentum, cont'd

- Cauchy's equation:

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} \right] = \rho \frac{D\vec{V}}{Dt} = \rho \vec{g} + \vec{\nabla} \cdot \sigma_{ij}$$

- σ_{ij} is not one of our primary unknowns.
- Primary unknowns are V, P, ρ, T



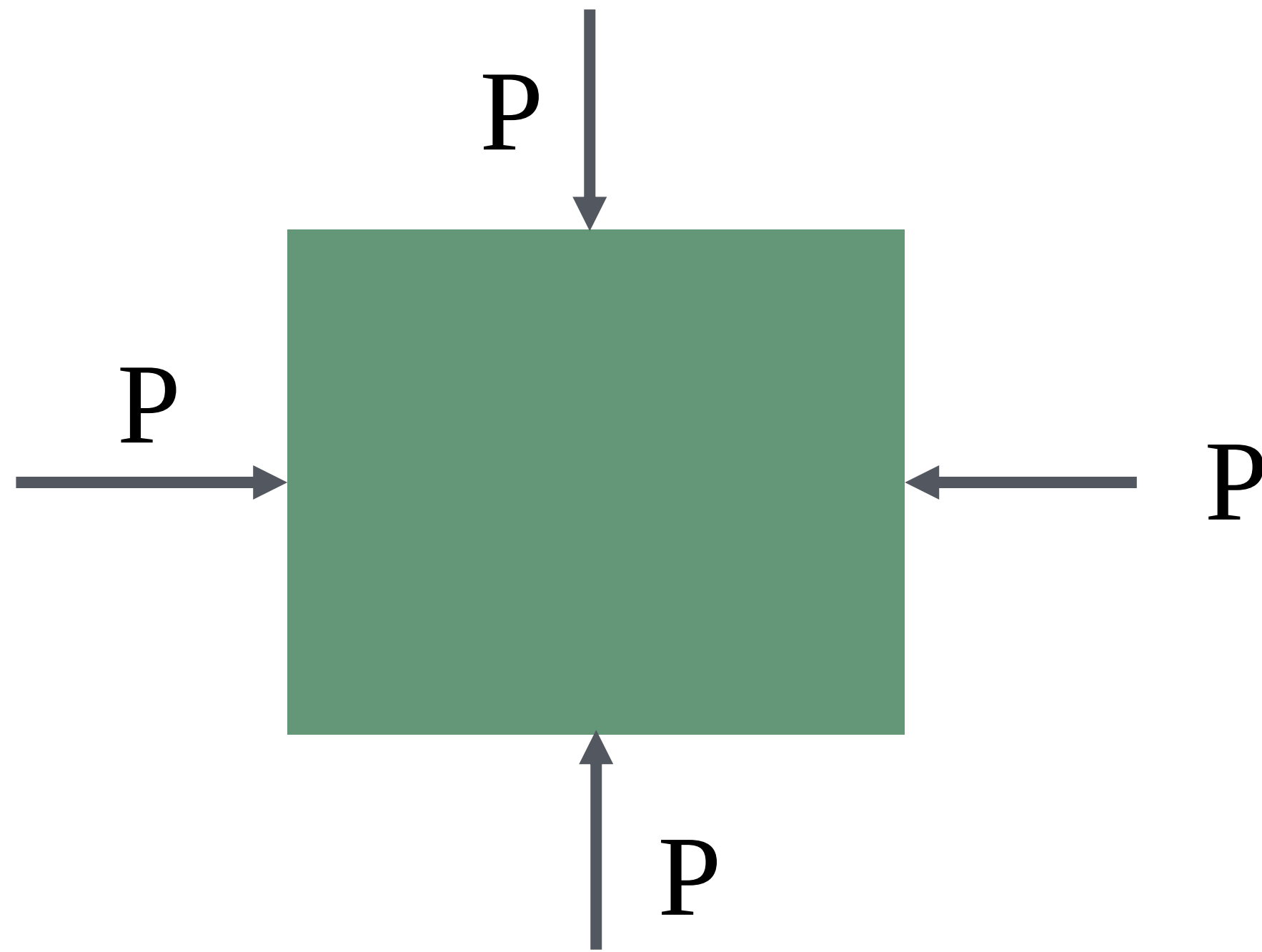
Goal

Express σ_{ij} as a function of the primary unknowns.



Stokes' assumptions for a Newtonian fluid

- σ_{ij} is at most a linear function of e_{ij} . [stress is a linear function of strain rate]
- Fluid is isotropic. [independent of direction]
- When $e_{ij} = 0$, σ_{ij} is non-zero and is equal to $-P\delta_{ij}$. [Only normal (hydraulic pressure) stress can exist.]



Wood grain texture (non-isotropic)



Navier-Stokes Equation

Fluid at rest:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix}$$

Moving fluids:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Mechanical pressure:

$$P_m = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Viscous stress tensor for an incompressible Newtonian fluid with constant properties:

$$\tau_{ij} = 2\mu\epsilon_{ij} \quad (9-55)$$



Navier-Stokes Equation, cont'd

$$\sigma_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 2\mu \frac{\partial v}{\partial y} & \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & 2\mu \frac{\partial w}{\partial z} \end{pmatrix}$$

X-momentum equation:

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho g_x + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + \mu \frac{\partial}{\partial z} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

Incompressible Navier-Stokes equation:

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} P + \rho \vec{g} + \mu \nabla^2 \vec{V}$$



The conservation equations

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

x-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

y-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

z-component of the incompressible Navier–Stokes equation:

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



Solving Navier-Stokes Equations

Step 1: Define coordinate system

Step 2: Determine appropriate boundary conditions (BC)

Step 3: Write out assumptions → most critical step; wrong/missing assumptions = wrong answer

Step 4: Cancel terms in continuity and N-S Equations based on assumptions

Step 5: Evaluate continuity equation. Cancel additional terms in N-S Eqns.

Step 6: Integrate remaining terms to get velocity equation (don't forget constant terms)

Step 7: Solve for constants using the boundary conditions (BC)



What assumptions do we typically use?

Continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Navier-Stokes Equations

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Typical Assumptions (situational)

- Steady
- Fully developed
- 2D (i.e., x and y only)
- 1 component (i.e., v=0) and w=0
- Neglect gravity



Assumptions cancel N-S terms (steady)

Continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Navier-Stokes
Equations

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



Assumptions cancel N-S terms (fully developed)

Continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Navier-Stokes
Equations

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



Assumptions cancel N-S terms (2D)

Continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Navier-Stokes
Equations

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



Assumptions cancel N-S terms (1 component $v=w=0$)

Continuity

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Navier-Stokes
Equations

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$



Example

- Consider steady, incompressible, laminar flow of a Newtonian fluid in the narrow gap between two infinite parallel plates (Fig. 9–55). The top plate is moving at speed V , and the bottom plate is stationary. The distance between these two plates is h , and gravity acts in the negative z -direction. There is no applied pressure other than hydrostatic pressure due to gravity. Calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate.

